4C.
CORRECTION OF A CONTRADICTION BETWEEN ELECTRODYNAMIC AND RELATIVISTIC ELECTROMAGNETIC MASS THEORIES (*)

"Correzione di una contraddizione tra la teoria elettrodinamica
e quella relativistica delle masse elettromagnetiche,"
Nuovo Cimento **25**, 159–170 (1923)

§ 1.– The theory of electromagnetic masses was studied for the first time by M. Abraham[1] before the discovery of the theory of relativity. Abraham therefore, as was natural, considered in his calculations the mass of a rigid system of charges in the sense of classical mechanics, and he found that, with the hypothesis that such a system had spherical symmetry, its mass varied with the speed and is precisely equal to[2] $\frac{4}{3}\frac{u}{c^2}$ (where u is the electrostatic energy of the system and c is the speed of light) for zero or very small speeds, but for speeds v comparable to c correction terms of order of magnitude v^2/c^2 appear which are a bit complicated. Even before the theory of relativity, Fitz Gerald introduced the hypothesis that solid bodies underwent a contraction in the direction of motion in the ratio

$$\sqrt{1-\frac{v^2}{c^2}}\ :1$$

and Lorentz redid Abraham's theory of electromagnetic masses, considering instead of rigid systems of electric charges in the sense of classical mechanics, systems that underwent this contraction. The result was that the rest mass, i.e., the limit of the mass for vanishing speed, was still $\frac{4}{3}\frac{u}{c^2}$, but the correction terms depending on v^2/c^2 changed. The experiences of Kaufmann, Bucherer and others with the mass of the β particles of radioactive bodies, and with high speed cathodic particles, decided in favor of the Lorentz theory, known as the contractile electron, against Abraham's theory of the rigid electron. This fact at the beginning was interpreted

*on the same argument see my notes in *rend. acc. lincei*, (5), 31, pp. 84, 306 (1922).

[1] Abraham, Theory of Electricity; Richardson, Electron Theory of Matter, Chapter XI; Lorentz, The Theory of Electrons, p. 37

[2] The electromagnetic mass of an homogeneous spherical shell of charge e, and radius r is $\frac{2}{3}\frac{e^2}{rc^2}$; but if we observe that the electrostatic energy is $u=\frac{1}{2}\frac{e^2}{r}$, we find the mass $\frac{4}{3}\frac{u}{c^2}$.

as a proof of the exclusively electromagnetic nature of the electron mass, because it was thought that otherwise their mass should be constant. Afterwards the discovery of the theory of relativity led to the consequence that all masses, electromagnetic or not, must vary with the speed like the mass of Lorentz's contractile electron; in this way the previous experiences left undecided the electromagnetic nature or not of the electron mass, being only a confirmation of the theory of relativity. On the other hand the special relativity theory first, and after the general theory, led to attribute to a system with energy u a mass u/c^2 and in this way arose a serious discrepancy between the Lorentz electrodynamic theory, which gives to a spherical distribution of electricity the rest mass $\frac{4}{3}\frac{u}{c^2}$, and special relativity which attributes to this distribution the mass u/c^2. That difference[3] is particularly serious given the great importance of the notion of the electromagnetic mass as a foundation for the electronic theory of matter.

This discrepancy showed up dramatically in two recent articles[4] in one of which, using the ordinary electrodynamic theory I considered the electromagnetic masses of a system with arbitrary symmetry, finding that in general they are represented by tensors instead of scalars, that reduce to $\frac{4}{3}\frac{u}{c^2}$ in the spherical symmetry case; in the other one instead, starting from general relativity, I considered the weight of the same systems which was in every case equal to $\frac{u}{c^2}\,g$, where g is the acceleration of gravity.

In the present work we will demonstrate precisely: that the difference between the two values of the mass obtained in the two ways originates in the concept of a rigid body in contradiction with the principle of relativity, which is applied in the electromagnetic theory (as well as in the contractile electron) and leads to the mass $\frac{4}{3}\frac{u}{c^2}$, while a better justified notion of rigid body conforming to the theory of relativity leads to the value u/c^2.

We note that the relativistic dynamics of the electron was done by M. Born[5] who starting from a point of view not essentially different from the usual one naturally found the rest mass $\frac{4}{3}\frac{u}{c^2}$.

Our considerations will be based on Hamilton's principle as the most suitable one to study a problem subject to very complicated constraints; in fact our system of electric charges must satisfy a constraint of a nature that is different from those considered in ordinary mechanics, since it has to exhibit, depending on its speed, the Lorentz contraction, as a consequence of the principle of relativity. To avoid misunderstandings, we note that while Lorentz contraction is of order v^2/c^2, its

[3]The experiences of Kaufmann and others cannot be useful to understand which of the two results is right, because these allow only the measurement of the speed dependent correction terms which are the same in both theories, while the difference is between the rest masses.
[4]E. Fermi, *N. Cim.*, VI, 22, pp. 176, 192 (1921).
[5]Max Born, *Ann. d. Phys.*, 30, p. 1 (1909)

influence on the electromagnetic mass is on the principal terms of this one, i.e., on the rest mass and therefore has a rather bigger importance, being appreciable for very small speeds as well.

§ 2.– So we consider a system of electric charges, sustained by a rigid dielectric that, under the action of an electromagnetic field generated partly from the system itself and partly from external sources, moves with a translation motion describing a time tube in the space-time.[6]

Let's explain what we mean by rigid translational motion. To do this we consider a Lorentz inertial frame and we suppose that in this frame at a certain time a point of the electric charge system has zero speed; we will say that the motion is translational if in the same frame at the same time all the other points of the system have zero speed. This fact is equivalent to saying that the time lines of our system points are trajectories orthogonal to a family of linear spaces; in fact in a Lorentz frame where the space axis is one of the spaces of the family and the time axis is perpendicular to it, the entire system is at rest at $t = 0$, because the space axis cuts orthogonally all the worldlines of all the points of the system. Using this definition of translational motion, which is substantially the one adopted by M. Born, the rigidity of the system is expressed by the fact that its figure in these spaces perpendicular to the tube remain invariable, i.e., all the sections of the tube are like each other.

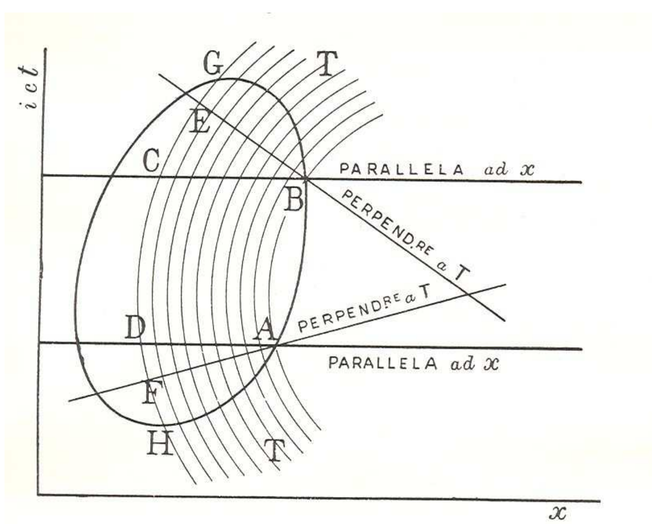

Figure 1. Translator note: "parallel to x" and "perpendicular to T".

[6]In the following we consider a Euclidean space-time, because we suppose that the considered electromagnetic fields are small enough to not modify the metric structure.

To be able to apply Hamilton's principle to our case we need a variation of our system movement consistent with the constraints of the problem, i.e., with the rigidity, correctly interpreted. Now we will show that the value $\frac{4}{3}\frac{u}{c^2}$ or $\frac{u}{c^2}$ is obtained for the electromagnetic mass, if we use either one variation or another that we will illustrate and distinguish from each other with the letters **A** and **B**. The variation **A**, however, as will immediately be clear, must be discarded because it is in contradiction with the principle of relativity. Let T be the time tube described by the system. In the figure the space (x, y, z) is represented by only one dimension along the x axis, and the time t is substituted by ict to have a definite metric.

Variation **A**: the variation that fulfils the rigidity constraint is an infinitesimal displacement, rigid in the ordinary kinematic sense, parallel to the space (x, y, z), of each section of the tube parallel to the space itself. In the figure we could obtain this variation by shifting each section t=const of the tube parallel to the x axis by an arbitrary infinitesimal segment. If we restrict ourselves to consider translational displacement, we will therefore have δx, δy, δz as arbitrary functions of only the time, and $\delta t = 0$.

Variation **B**: the variation that fulfils the rigidity constraint is an infinitesimal displacement perpendicular to the tube of each section normal to the tube itself, rigid in the ordinary kinematic sense. In the figure we could obtain this variation shifting, by an arbitrary segment, each normal section of the tube parallel to itself.

Among these two variations **A** is in obvious contradiction with the principle of relativity and must be discarded because, not even being Lorentz invariant, it depends on the particular frame (t, x, y, z) we have chosen and can't be the expression of any physical notion, like rigidity. The variation **B** instead, besides satisfying Lorentz invariance, since it only consists of elements of the tube T completely independent of the position of the frame axes, is the only one presents itself naturally, like that based on a rigid virtual displacement in the frame where at the instant considered the system of charges has zero speed. Now it would be wrong to think that the difference between the consequences of the two methods of variation **A** and **B** is significant only for high speeds, i.e., when the tube T has a big slope with respect to the time axis. Instead the calculations we are going to develop will demonstrate immediately that the difference is felt already at zero speed and that precisely **A** gives $\frac{4}{3}\frac{u}{c^2}$ as the electromagnetic mass the while **B** gives instead u/c^2.

§ 3.– We indicate the coordinates of time and space by (t, x, y, z) or (x_0, x_1, x_2, x_3) as convenient and let ϕ_i be the four-potential and

$$F_{ik} = \frac{\partial \phi_i}{\partial x_k} - \frac{\partial \phi_k}{\partial x_i}$$

the electromagnetic field, and **E** and **H** the electric and magnetic forces that derive from it.

Hamilton's principle that summarizes the laws of Maxwell Lorentz and those of mechanics says that:[7] the total action, i.e., the sum of the actions of the electromagnetic field and of the material and electric masses, has zero variation under the effect of an arbitrary variation of the ϕ_i and of the coordinates of the points of the electric charge world lines that respect the constraints and are zero on the boundary of the integration region. In our case there aren't material masses, and the only variable elements are the coordinates of the points on the world lines of the charges; therefore it is enough to consider only the action of the electric charges, i.e.:

$$W = \sum_i \int de \int \phi_i \, dx_i$$

where de is the generic element of electric charge and the second integral is calculated on the timeline arc described by de that is contained in the four-dimensional region G of integration. For each system of variations δx_i satisfying the constraints and that *vanishes on the boundary of G*, one must have $\delta W = 0$, i.e.:

$$\sum_{ik} \int \int de \; F_{ik} \delta x_i dx_k = 0 \; , \tag{1}$$

Now we must examine separately the results obtained substituting δx_i by the values given by the system of variations **A** or **B**.

§ 4.– *Consequences of the system of variations* A. — In this case the region of integration reduces to ABCD. The regions BCG, ADH give no contribution, because in them all the δx_i are zero since they have to vanish on the boundary of G, and therefore along the curves BG, AH and must be constants for t =const, i.e., on the straight lines parallel to the x axis. If we label the times of A and B by t_1 and t_2, the equation (1) can be written, since $\delta t = 0$ and δx, δy, δz are functions of the time only:

$$\sum_{ik} \int_{t_1}^{t_2} dt \; \delta x_i \int de \; F_{ik} \frac{dx_k}{dt} \qquad (i = 1, 2, 3) \qquad (k = 0, 1, 2, 3) \; .$$

Since the δx_i are arbitrary functions of t, we obtain the three equations

$$\int de \sum_k F_{ik} \frac{dx_k}{dt} = 0$$

i.e.,

$$\int de [E_x + \frac{dy}{dt} H_z - \frac{dz}{dt} H_y] = 0 \qquad \text{and the analogous two.}$$

If at the chosen instant the system has zero speed in the frame (t, x, y, z) the three equations can be summarized by a single vector equation:

$$\int \mathbf{E} \, de = 0 \; . \tag{2}$$

[7] Weyl, *Space, Time, Matter*, pp. 194–196; Berlin, Springer (1921).

We could have obtained this equation without calculations if, as is usually done in the ordinary treatment and as M. Born essentially does in the cited work, we had set to zero from the beginning the total force acting on the system. We wanted deduce it using Hamilton's principle to show the fault of its origin, since it follows from the system of variations **A** that it is in contradiction with the relativity principle. From (2) follows immediately the value $\frac{4}{3}\frac{u}{c^2}$ for the electromagnetic mass. Suppose in fact that $\mathbf{E}$ is the sum of a part $\mathbf{E}^{(i)}$ due to the system itself, plus a uniform field $\mathbf{E}^{(e)}$ due to external sources. (2) gives:

$$\int \mathbf{E}^{(i)}\, de + \int \mathbf{E}^{(e)}\, de = 0 \ .$$

Now $\int de = e =$ charge; and then $\mathbf{E}^{(e)} \int de = \mathbf{F} =$ external force. In the spherical symmetry case, both direct calculation, and the well known considerations of the electromagnetic moment[8] show that:

$$\int \mathbf{E}^{(i)}\, de = -\frac{4}{3}\frac{u}{c^2}\mathbf{\Gamma} \ ,$$

where $\mathbf{\Gamma}$ is the acceleration.

The previous equation then becomes:

$$\mathbf{F} = \frac{4}{3}\frac{u}{c^2}\mathbf{\Gamma}$$

that compared to the fundamental law of point dynamics, $\mathbf{F} = m\mathbf{\Gamma}$, gives:

$$m = \frac{4}{3}\frac{u}{c^2} \ .$$

§ 5.– *Consequences of the system of variations* B. — In this case the same considerations of the previous section demonstrate that the region of integration reduces to ABEF, i.e., to the region bounded by two normal sections of the tube T. By the use of infinite normal sections Decomposing it using an infinite number of normal sections into layers of infinitesimal thickness, and in order to calculate the contribution of one of these to the integral (1) we refer to its rest frame, by considering the space (x,y,z) parallel to the layer. For this $\delta t = 0$ will hold, while δx, δy, δz will be arbitrary constants. Moreover $dx = dy = dz = 0$, because the speed of all the points is zero, $dt =$ height of the layer, that will vary for each point, because the layer has for its faces two normal sections which in general are not parallel. If O is a generic point but fixed in the layer, for example the origin of coordinates, in which dt has the value dt_0, and $\mathbf{K}$ is the vector with the orientation of the principal normal to the timeline passing for O and size equal to its curvature, we have manifestly, since dt is the thickness at the generic point P of the layer:

$$dt = dt_0[1 - \mathbf{K} \cdot (P - O)] \ .$$

[8]RICHARDSON loc. cit.

Since the speed is zero we have

$$\mathbf{K} = -\mathbf{\Gamma}/c^2 \ ,$$

and therefore:

$$dt = dt_0 \left(1 + \frac{\mathbf{\Gamma} \cdot (P - O)}{c^2} \right) \ .$$

Substituting these values we find that the contribution of our layer to the integral (1) is:

$$-dt_0 \Big\{ \delta x \int \Big(1 + \frac{\mathbf{\Gamma} \cdot (P - O)}{c^2} \Big) E_x de + \delta y \int \Big(1 + \frac{\mathbf{\Gamma} \cdot (P - O)}{c^2} \Big) E_y de + \\ + \delta z \int \Big(1 + \frac{\mathbf{\Gamma} \cdot (P - O)}{c^2} \Big) E_z de \Big\} \ .$$

This expression must vanish for all the values of δx, δy, δz and we obtain from it three equations that can be summarized in the single vector equation:

$$\int \Big(1 + \frac{\mathbf{\Gamma} \cdot (P - O)}{c^2} \Big) \mathbf{E} \, de = 0 \tag{3}$$

A correct application of Hamilton's principle has then brought us to (3) instead of (2). Now it's easy to examine the consequences. Setting

$$\mathbf{E} = \mathbf{E}^{(i)} + \mathbf{E}^{(e)}$$

we find

$$\int \mathbf{E}^{(i)} de + \int \mathbf{E}^{(i)} \ \frac{\mathbf{\Gamma} \cdot (P - O)}{c^2} de + e \mathbf{E}^{(e)} + \mathbf{E}^{(e)} \int \frac{\mathbf{\Gamma} \cdot (P - O)}{c^2} de = 0 \ .$$

In the spherical symmetry case we have as before

$$\int \mathbf{E}^{(i)} \ de = -\frac{4}{3} \frac{u}{c^2} \mathbf{\Gamma} \ ;$$

substituting in the previous equation we find that $\mathbf{E}^{(e)}$ is compared only with the terms that contain $\mathbf{\Gamma}$. If we neglect the $\mathbf{\Gamma}^2$ terms[9], we can neglect the last integral, and we obtain:

$$-\frac{4}{3} \frac{u}{c^2} \mathbf{\Gamma} + \int \mathbf{E}^{(i)} \ \frac{\mathbf{\Gamma} \cdot (P - O)}{c^2} de + \mathbf{F} = 0 \ . \tag{4}$$

To calculate the integral which appears in (4) we observe that $\mathbf{E}^{(i)}$ is the sum of the Coulomb force

$$= \int \frac{P - P'}{r^3} de'$$

[9]More precisely the number compared to which the quadratic terms are negligible is $\Gamma \ell / c^2$, where ℓ is the largest length which appears in the problem. It is clear that such an approximation is more than justified in common situations.

(P' is the point of charge de' and $r = \overline{PP'}$), and of a term containing $\mathbf{\Gamma}$ that can be neglected because it would give a contribution containing Γ^2. Our integral then becomes:

$$\int\int \frac{P-P'}{r^3}\,\frac{\mathbf{\Gamma}\cdot(P-O)}{c^2} de\, de' \ ;$$

or exchanging P with P', which doesn't change matters, and taking the half sum of the two values obtained in this way:

$$\frac{1}{2}\int\int \frac{P-P'}{cr^3}\,[\mathbf{\Gamma}\cdot(P-P')] de\, de' \ .$$

We observe that, in our approximation $\mathbf{\Gamma}$ is constant for all the points and then can be taken out of the integrals. Therefore the x component of the previous integral is:

$$\frac{1}{2c^2}\Big\{\mathbf{\Gamma}_x \int\int \frac{(x-x')^2}{r^3} de\, de' + \mathbf{\Gamma}_y \int\int \frac{(y-y')(x-x')}{r^3} de\, de' + \\ + \mathbf{\Gamma}_z \int\int \frac{(z-z')(x-x')}{r^3} de\, de'\Big\} \ .$$

Now, since the system has spherical symmetry, to each segment PP' corresponds an infinite number of other segments differing only in orientation. In the three integrals we can therefore substitute

$$(x-x')^2, (x-x')(y-y'), (x-x')(z-z')$$

by their average values for all the possible orientations of PP', which are; $\frac{1}{3}r^2$, 0, 0.

With that the x component becomes:

$$\frac{\Gamma_x}{3c^2}\frac{1}{2}\int\int \frac{de\, de'}{r}$$

We now observe that the expression

$$\frac{1}{2}\int\int \frac{dede'}{r}$$

is the electrostatic energy u; going back to vector notation we find for the integral appearing in equation (4) the expression: $\dfrac{u}{3c^2}\mathbf{\Gamma}$. (4) becomes in this way:

$$\frac{u}{c^2}\mathbf{\Gamma} = \mathbf{F} \qquad (5)$$

that says the electromagnetic mass is u/c^2.